Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

SOLUTIONS OF EXERCISES.

162

A BODY is projected at an angle of 30° with the horizon, with a given velocity. Determine the constant resistance it must suffer in the direction contrary to its motion in order that it may come to rest when it returns to the horizontal plane whence it started. Also determine the horizontal range, time of flight, and length of trajectory.

[*Jas. M. Ingalls.*]

SOLUTION.

Let V be the initial velocity, a the angle of projection, φ the inclination at a point of the trajectory where the velocity is v, U and u the corresponding *horizontal* velocities, r the constant resistance, and w the weight of the projectile. Put $r/w = n$. Then we have

$$\frac{du}{u} = n\frac{d\varphi}{\cos\varphi};$$

whence

$$\log u = n \log \tan\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] + C.$$

Let v_0 be the velocity when $\varphi = 0$, that is, at the summit of the trajectory; then $C = \log v_0$, and we have

$$u = v_0 \tan^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right].$$

$$\therefore v_0 = u \cot^n\left[\frac{1}{4}\pi + \frac{1}{2}\varphi\right] = U \cot^n\left[\frac{1}{4}\pi + \frac{1}{2}a\right]. \tag{1}$$

Substituting the above value of u in the differential equations

$$dt = -\frac{u}{g}\sec^2\varphi d\varphi,$$

$$dx = -\frac{u^2}{g}\sec^2\varphi d\varphi,$$

$$dy = -\frac{u^2}{g}\tan\varphi \sec^2\varphi d\varphi,$$

$$ds = -\frac{u^2}{g}\sec^3\varphi d\varphi,$$

The rise, range, time of flight, and length of trajectory are to be determined from the formulæ

$$Y=\frac{1}{g}\int_0^a v^2\tan\varphi\,d\varphi\,, \quad (1) \qquad T=\frac{1}{g}\int_{-\frac{1}{2}\pi}^a v\sec\varphi\,d\varphi\,, \quad (3)$$

$$X=\frac{1}{g}\int_{-\frac{1}{2}\pi}^a v^2\,d\varphi\,, \quad (2) \qquad S=\frac{1}{g}\int_{-\frac{1}{2}\pi}^a v^2\sec\varphi\,d\varphi\,; \quad (4)$$

in which the values of v as determined above in terms of φ must be substituted.

There are two ways offered for determining r: we may put $-\frac{1}{2}\pi$ for the lower limit in the integral (1) instead of 0 and equate the result to zero and solve for r; otherwise equate the values of S in (4) to that in $S=V^2/2r$ and solve the result for r. Either method gives $r=\frac{5}{4}g$.

Thus by the former, put $2r/g=n$, $\frac{1}{4}\pi+\frac{1}{2}a=\beta$, and $\frac{1}{4}\pi+\frac{1}{2}\varphi=\theta$ for brevity; then

$$\int_{-\frac{1}{2}\pi}^a \tan^{\frac{2r}{g}}(\tfrac{1}{4}\pi+\tfrac{1}{2}\varphi)\sec^2\varphi\tan\varphi\,d\varphi=0$$

becomes

$$\int_0^{\tan\beta}(\tan^{n-3}\theta-\tan^{n+1}\theta)\,d\,(\tan\theta)=0\,,$$

or

$$\frac{\tan^{n-2}\beta}{n-2}=\frac{\tan^{n+2}\beta}{n+2}\,.$$

Whence

$$n=2\,\frac{\tan^4\beta+1}{\tan^4\beta-1}\,.$$

This gives $r=\frac{5}{4}g$, since $a=30°$ and $\beta=60°$.

$$\therefore\ S=\frac{2}{5}\,\frac{V^2}{g}\,.$$

From the same integral we have for the rise, putting

$$K=\frac{V^2}{g}\,\frac{\cos^2 30°}{\tan^{\frac{5}{2}}60°}=\frac{1}{4\sqrt[4]{3}}\,\frac{V^2}{g}$$

for shortness,

$$Y=\frac{1}{g}\int_0^a v^2\tan\varphi\,d\varphi=-\frac{1}{4}K\int_1^{\sqrt{3}}(\tan^{-\frac{1}{2}}\theta-\tan^{\frac{7}{2}}\theta)\,d\,(\tan\theta)$$

$$=-\frac{1}{4}K\left\{2\tan^{\frac{1}{2}}\theta-\frac{2}{9}\tan^{\frac{9}{2}}\theta\right\}_1^{\sqrt{3}}=\frac{1}{9\sqrt[4]{3}}\,\frac{V^2}{g}\,.$$

In like manner

$$X=\frac{1}{g}\int_{-\frac{1}{2}\pi}^a v^2 d\varphi=\frac{1}{2}K\int_0^{\sqrt{3}}(\tan^{\frac{1}{2}}\theta+\tan^{\frac{5}{2}}\theta)\,d\,(\tan\theta)$$

$$=K\left\{\frac{1}{3}\tan^{\frac{3}{2}}\theta+\frac{1}{7}\tan^{\frac{7}{2}}\theta\right\}_0^{\sqrt{3}}=\frac{4\sqrt{3}}{21}\,\frac{V^2}{g}\,.$$

Finally

$$T = \frac{1}{g}\int_{-\frac{1}{2}\pi}^{\alpha} v \sec\varphi \, d\varphi = \frac{V}{g}\frac{\cos 30^\circ}{\tan^{\frac{5}{4}} 60^\circ}\int_{-\frac{1}{2}\pi}^{\alpha} \tan^{\frac{5}{4}}(45^\circ + \tfrac{1}{2}\varphi) \sec^2\varphi \, d\varphi$$

$$= \frac{1}{3^{\frac{5}{8}}}\frac{V}{4g}\int^{\sqrt{3}}(\tan^{\frac{5}{4}}\theta + \tan^{\frac{13}{4}}\theta)\, d\,(\tan\theta) = \frac{44\sqrt{3}}{153}\,\frac{V}{g}.$$

[*W. H. Echols.*]

247

Two sides of a triangle are a and b; find the average length of the third side. [*Artemas Martin.*]

SOLUTION.

I. Supposing the third side to vary uniformly, let x = the third side; then it is obvious that $a + b$ is the *greatest* value x can have, and $a - b$ is the *least* value x can have; and $\frac{1}{2}[(a + b) + (a - b)] = a$ is the average length of the third side. *Otherwise.* Let x = third side, and $\varDelta$ = its average length; then

$$\varDelta = \int_{a-b}^{a+b} x\,dx \div \int_{a-b}^{a+b} dx, = a, \text{ as before.}$$

II. Supposing the angle included by the two given sides to vary uniformly, let φ be that angle; then $x = \sqrt{(a^2 + b^2 - 2ab\cos\varphi)}$, and

$$\varDelta = \int_0^{\pi} \sqrt{(a^2 + b^2 - 2ab\cos\varphi)}\, d\varphi \div \int_0^{\pi} d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi} \sqrt{(a^2 + b^2 - 2ab\cos\varphi)}\, d\varphi.$$

Let $\varphi = \pi - 2\theta$, then $\cos\varphi = -\cos 2\theta = 2\sin^2\theta - 1$, $d\varphi = -2d\theta$, and by substitution

$$\varDelta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{(a^2 + b^2 + 2ab - 2ab\sin^2\theta)}\, d\theta$$

$$= \frac{2(a+b)}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{\left[1 - \frac{2ab}{(a+b)^2}\sin^2\theta\right]}\, d\theta$$

$$= \frac{2(a+b)}{\pi}\,\mathrm{E}\left[\frac{\sqrt{2ab}}{a+b}, \tfrac{1}{2}\pi\right].$$

[*Artemas Martin.*]

289

The points P, Q, R, S, T, U are the vertices of a regular hexagon inscribed in a circle with centre O. Join PR, cutting OQ in A; AS, cutting OR in B; BT, cutting OS in C; etc. Prove that OA, OB, OC, OD, etc. are proportional to the reciprocals of the natural numbers 1, 2, 3, 4, etc. [*Yale.*]

SOLUTION.

If $OP = R$, $OA = \frac{1}{2}R$. The triangles, AOB and RBS are similar; $\therefore OB = \frac{1}{2}BR = \frac{1}{3}R$. The triangles OBC and TCS are similar and $OB = \frac{1}{3}R = \frac{1}{3}ST$; $\therefore OC = \frac{1}{3}CS = \frac{1}{4}R$. And so on.

$\therefore OP : OA : OB : OC : \ldots :: 1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \ldots$ [*W. B. Richards.*]

293

If a triangle be cut out of paper and doubled over so that the crease passes through the centre of the circumscribing circle and the angle A, the area of the triangular double-part will be

$$\tfrac{1}{2}b^2 \sin^2 C \cos C \operatorname{cosec}(2C - B) \sec(C - B);$$

the angle C being greater than B. [*Yale.*]

SOLUTION.

Call O the centre of the circumscribed circle OA cuts BC in D. Make $\angle ADF = \angle ADC$. DF cuts AB in F. Take DE on $DC = DF$. Then DAE is the triangle required.

We know that $\angle ADE = 90° - (C - B)$, and $\angle BAD = DAE = 90° - C$; also

$$\text{area } DAE = \tfrac{1}{2}\,\frac{\overline{AD}^2 \sin DAE \sin ADE}{\sin(DAE + ADE)}.$$

But $AD = \dfrac{b \sin C}{\sin ADE}$;

$$\therefore \text{area } DAE = \tfrac{1}{2}\,\frac{b^2 \sin^2 C \sin DAE}{\sin ADE \sin(DAE + ADE)}$$

$$= \tfrac{1}{2}b^2 \sin^2 C \cos C \sec(C - B) \operatorname{cosec}(2C - B).$$

[*T. U. Taylor.*]

* * *

EXERCISE.

320

A hollow sphere, external and internal radii R and r, and specific gravity s, is partly filled with water, and floats in a pond, the water in the sphere being on a level with the surface of the pond. Find the quantity of water in the hollow sphere. [*Artemas Martin.*]